Application of Edge-chain Matrices of Graph to find all Eulerian Cycles

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Abstract: The initial edge-chain matrix and general edge-chain matrix of graph are presented. The operations of the general edge-chain matrices are derived, by which a method to find all Eulerian cycles is obtained. Only through some power operations of the initial edge-chain matrix, can reveal all Eulerianian cycles which are showed in the final edge-chain matrix. This method can determine whether Eulerianian cycles exist or not and if they do can also find out all of them. It is effective to directed or undirected finite graph. And it can be simplified by computations of some row vectors and column vectors of some power of the initial edge-matrix. This pure mathematical method shows the results more intuitive and makes program operation easier.

1. Introduction

In graph theory, an Eulerian path is a path that visits each edge exactly once. If such a path exists, the graph is called semi-eulerian. An Eulerian cycle or Eulerian circuit is an Eulerian path which starts and ends on the same vertex. If such a cycle exists, the graph is called Eulerian [1]. These problems first discussed by Leonhard Euler while solving the famous Seven Bridges of Konigsberg problem in 1736. Euler proved the necessary condition for the existence of Eulerian cycles [1].

The number of Eulerian cycles in digraphs can be calculated by using the so-called BEST theorem [2]. While counting the number of Eulerian cycles on undirected graphs is much more difficult. This problem is known to be #P-complete [3]. In a positive direction, a Markov chain Monte Carlo approach [4], via the Kotzig transformations is believed to give a sharp approximation for the number of Eulerian cycles [5-6], though as yet there is no proof of this fact (even for graphs of bounded degree). There are also some special algorithms to count the numbers of Eulerian cycles for some special graphs, such as asymptotics, asymptotic enumeration, etc, e.g. [7-14]. All these existing algorithms and their improvements mainly focus on the realization of algorithms and their time complexities.

This paper mainly presents two new concepts, initial edge-chain matrix and general edge-chain matrix, and then we define some operations to study the Eulerian graph problem. This mathematical method can judge whether the Eulerian cycle (path) exists in a graph or not. And if it exists, can find all Eulerian cycles (paths) which are showed in the final edge-matrix.

Finite graphs (directed or undirected) are discussed in this paper.

2. Edge-chain matrices of a graph and their addition operation

Let $G (V, E)$ be a labeled graph with V as the set of vertices and E the set of edges. Generally let $V = \{V_1, V_2, \dots, V_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$ separately, we denote simply $E = \{1, 2, \dots, m\}$ in this paper.

Definition 1**.** In a given graph (directed or undirected graph), for any two vertexes, a list of different edges which connect them is called an edge-chain between these two points. The number of edges contained in an edge-chain is called the length of an edge-chain.

Definition 2. In a given graph, if an edge-chain starts and ends at a same point, then it is called an edge-chain cycle.

An edge-chain cycle is an Eulerian cycle when it contains every edge of a graph.

We use numbers to express the corresponding edges, and a permutation of several numbers to express an edge-chain containing these edges. For example, 2135 means the edge-chain $\{e_2, e_1, e_3, e_5\}$ or directly {2, 1, 3, 5} in this paper. Sometimes there are several edge-chains from vertex *i* to vertex *j*. We use the symbol"$\oplus$" to indicate the operation of "addition" (union operation actually) of these different edge-chains. For example, $(245 \oplus 2135)$ indicates that there are two edge-chains {2, 4, 5} and {2, 1, 3, 5} between the certain two vertexes.

Definition 3. Let *G (V, E)* be a labeled graph with *V* as the set of vertices and *E* the set of edges. And let *n* = |*V*|. The matrix $\mathcal{P} = (p_{ij})_{n \times n}$ is called an initial edge-chain matrix, where

$$p_{ij} = \begin{cases} one\ or\ several\ edges\ from\ vertex\ i\ to\ vertex\ j \\ 0;\ \ if\ \ a\ edge\ is\ nonexistent\ from\ vertex\ i\ to\ vertex\ j \end{cases}$$

Definition 4: Let *G (V, E)* be a labeled graph with *V* as the set of vertices and *E* the set of edges. And let *n* = |*V*|. The matrix $\mathcal{Q} = (q_{ij})_{p \times q}$ $(1 \le p, q \le n)$ is called a general edge-chain matrix, edge-chain matrix in short, where

$$q_{ij} = \begin{cases} some\ edge\text{-}chains\ from\ vertex\ i\ to\ vertex\ j\ (under\ a\ certain\ condition); \\ 0;\ \ if\ \ a\ edge\text{-}chain\ is\ nonexistent\ from\ vertex\ i\ to\ vertex\ j\ (under\ a\ certain\ condition). \end{cases}$$

Obviously, a general edge-chain matrix is an initial edge-chain matrix when $q_{ij}=p_{ij}$ and $p=q=n$.

It should be noticed that a general edge-chain matrix is a general $p \times q$ matrix.

For ease of presentation, all matrices are edge-chain matrices in this paper.

Definition 5: Let matrix $\mathcal{Q} = (q_{ij})$ be an edge-chain matrix of a graph. If an element q_{ij} contains some different edge-chains, then the maximum length of these edge-chains is called the length of q_{ij}. The length of element 0 is 0.The maximum length of all elements in the matrix $\mathcal{Q}$ is called the length of matrix $\mathcal{Q}$.

For example, when q_{25}=$(145 \oplus 2136)$, the length of q_{25} is 4.

Obviously, the length of the initial edge-chain matrix is 1 and the length of any edge-chain matrix is no bigger than m.

When an element in edge-chain matrix contains some different edge-chains, we rank these edge-chains by their lengths from small to big at first, and if they have the same length then rank them in lexicographical order, for example q_{25}=$(145 \oplus 2136 \oplus 2145)$.

Definition 6. The reverse permutation of an edge-chain is called an inverse order of the edge-chain. When both an edge-chain and its inverse order are showed at a same expression, they should be regarded as the same edge-chain, and be expressed by the first according to the lexicographical order.

For example, the reverse order of 2135 is 5312, and $2135 \oplus 5312 = 2135$.

Definition 7. If all elements in a edge-chain matrix are 0, then the matrix is called 0 edge-chain matrix, denoted as Θ.

Definition 8. (Edge-chain matrices addition). Let $\mathcal{F} = (f_{ij})$ and $\mathcal{G} = (g_{ij})$ are two $p \times q$ edge-chain matrices, the addition of these two matrices is defined as $\mathcal{F} \oplus \mathcal{G} = (f_{ij} \oplus g_{ij})$.

We stipulate that for any f_{ij} and g_{ij}, there are $f_{ij} \oplus 0 = f_{ij}, 0 \oplus g_{ij} = g_{ij}, f_{ij} \oplus f_{ij} = f_{ij}$, $0 \oplus 0 = 0$.

The addition of the two edge-chain matrices can be extended to the case of multiple edge-chain matrices, and such addition satisfies the commutative law and associative law.

(1) Commutativity for summation: $\mathcal{F} \oplus \mathcal{G} = \mathcal{G} \oplus \mathcal{F}$.

(2) Associativity for summation: $(\mathcal{F} \oplus \mathcal{G}) \oplus \mathcal{H} = \mathcal{F} \oplus (\mathcal{G} \oplus \mathcal{H})$.

3. Multiplication of edge-chain matrices

Definition 9. (Edge-chain multiplication). Let $\mathcal{F} = (f_{ij})$ and $\mathcal{G} = (g_{ij})$ are two edge-chain matrices, we define the product (or multiplication) of two elements f_{ik} and g_{kj} is the connected edge-chains of these two edge-chains, denoted as $h_{ij} = f_{ik} \otimes g_{kj}$. If the connection is not an edge-chain (means the connection contains at least two same edges), then record it as 0. And if all connections are not edge-chains, then let $h_{ij} = 0$. In the case that both f_{ik} and g_{kj} contain several edge-chains, then their product can be expanded as the operation of polynomial multiplication.

For example, if f_{25}=$(145 \oplus 2136 \oplus 2145)$ and q_{53}=$(34 \oplus 637)$, then

$$f_{25} \otimes q_{53} = (145 \oplus 2136 \oplus 2145) \otimes (34 \oplus 637)$$
$$= (145637 \oplus 2145637).$$

We stipulate that for any f_{ik} and g_{kj}, there are $f_{ik} \otimes 0 = 0 \otimes g_{kj} = 0 \otimes 0 = 0$.

It is easy to verify that the edge-chain multiplication satisfies the following rules.

(3) Dispensability:

$$(f_{ik} \oplus g_{ik}) \otimes h_{kj} = (f_{ik} \otimes h_{kj}) \oplus (g_{ik} \otimes h_{kj});$$
$$f_{ik} \otimes (g_{kj} \oplus h_{kj}) = (f_{ik} \otimes g_{kj}) \oplus (f_{ik} \otimes h_{kj}).$$

Definition 10 (Edge-chain matrix multiplication). Let $\mathcal{F} = (f_{ij})$ be a $p \times s$ edge-chain matrix and $\mathcal{G} = (g_{ij})$ an $s \times q$ edge-chain matrix. We define the multiplication of matrix $\mathcal{F}$ and $\mathcal{G}$ is a $p \times q$ edge-chain matrix $\mathcal{H} = (h_{ij})$, denoted as $\mathcal{F} \otimes \mathcal{G} = \mathcal{H}$, where

$$h_{ij} = (f_{i1} \otimes g_{1j}) \oplus (f_{i2} \otimes g_{2j}) \oplus \cdots \oplus (f_{is} \otimes g_{sj}), (i = 1, 2, 3, \cdots, p;\ j = 1, 2, \cdots, q).$$

It is easy to prove, as in the case of ordinary matrix multiplication, the multiplication of the edge-chain matrices satisfies the associative law. So let $\mathcal{F}, \mathcal{G}, \mathcal{R}$ be three edge-chain matrices and assume that the multiplication of these matrixes are legal, we have

(4) Associativity for multiplication:

$$(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{R} = \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{R}).$$

Definition 11. Let $\mathcal{F}$ be a square edge-chain matrix. For any positive integer *k (k>1)*, define $\mathcal{F}^k = \mathcal{F}^{k-1} \otimes \mathcal{F}$.

When $\mathcal{F} = \mathcal{P}$, we have $\mathcal{P}^k$ for any positive integer *k*.

4. The longest chains and all Eulerian cycles

Theorem 1. Let the matrix $\mathcal{P} = (p_{ij})_{n \times n}$ be the initial edge-chain matrix of a graph, then the length of matrix $\mathcal{P}^2$ is 2 or 0; the length of matrix $\mathcal{P}^3$ is 3 or 0, and so on. But there exists the smallest positive integer l $(1 \le l \le m)$ such that $\mathcal{P}^l \ne \Theta$ and $\mathcal{P}^{l+1} = \mathcal{P}^{l+2} = \cdots = \Theta$. So the length of the matrix $\mathcal{P}^l$ is l.

Proof. Obviously, the length of $\mathcal{P}$ is 1. By the definition 9-10, the lengths of all entries in $\mathcal{P}^2$ are 0 and 2. If the length of all entries in $\mathcal{P}^2$ is 0, then $\mathcal{P}^2 = \Theta$ and $m = 1$, otherwise the length of the matrix $\mathcal{P}^2$ is 2 and $\mathcal{P}^2 \ne \Theta$. Similarly, if $\mathcal{P}^2 \ne \Theta$, then the lengths of all entries in $\mathcal{P}^3$ are 0 and 3. If the length of all entries in $\mathcal{P}^3$ is 0, then $\mathcal{P}^3 = \Theta$ and $m = 2$, otherwise the length of the matrix $\mathcal{P}^3$ is 3 and $\mathcal{P}^3 \ne \Theta$, and so forth. With the increase of the power, the length of edge-chain between any two vertexes increases monotonously. However, this is a finite graph, the length of edge-chain between any two vertexes is limited and no more than *m*. So there exists the smallest positive integer l $(1 \le l \le m)$ such that $\mathcal{P}^l \ne \Theta$ and $\mathcal{P}^{l+1} = \mathcal{P}^{l+2} = \cdots = \Theta$. Thus, the length of the edge-chain matrix $\mathcal{P}^l$ is l. □

Definition 12. Let matrix $\mathcal{P}=(p_{ij})_{n\times n}$ be the initial edge-chain matrix of a graph. If there exists a smallest positive integer l $(l\geq 1)$ such that $\mathcal{P}^l\neq\Theta$ and $\mathcal{P}^{l+1}=\Theta$, we call l the power length of the matrix $\mathcal{P}$.

By definition 12 and theorem 1, the power length of the initial edge-chain matrix $\mathcal{P}$ is the length of the longest edge-chain. The length of each element in the matrix $\mathcal{P}^l$ is either l or 0. So the edge-chains with the length l are all the longest edge-chains.

For any two vertexes, there also exists the longest edge-chain. If we find the longest edge-chain from the vertex i to vertex j, then let $\mathcal{P}_i$ be the row i of the matrix $\mathcal{F}$ and it is continuously multiplied by matrix $\mathcal{P}$ until the element of j- th column of the final product is 0, so the element of j- th column of the former step product which is not 0 corresponds the longest edge-chain between these two points.

Corollary 1. If the power length of the initial edge-chain matrix $\mathcal{P}=(p_{ij})_{n\times n}$ of a graph is l, then the length of matrix $\mathcal{P}^k$ is k for any positive integer k $(1\leq k\leq l)$. That is, there exists edge-chains with length k in matrix $\mathcal{P}^k$.

From Corollary 1, if we find all edge-chains with length k from the vertex i to vertex j, then let $\mathcal{P}_i^{(1)}$ be the row i of the matrix $\mathcal{P}$ and multiplied by matrix $\mathcal{P}$ $k-1$ times. If the element of j th column of the final product is 0, then there does not exist edge-chains with length k; and if it is not 0, then it is all edge-chains with length k.

Corollary 2. Suppose the power length of the initial edge-chain matrix $\mathcal{P}=(p_{ij})_{n\times n}$ of a graph is l $(1<l<m)$, if the element *(i,j)* in the sum matrix $\sum_{k=1}^{l}\mathcal{P}^k=\mathcal{P}\oplus\mathcal{P}^2\oplus\cdots\oplus\mathcal{P}^m$ is 0, then the two vertexes from the vertex i to vertex j are not connective; and if it is not 0, then they are connective and the element *(i,j)* gives all edge-chains between these two vertexes. If all elements except the diagonal in the sum matrix $\sum_{k=1}^{l}\mathcal{P}^k=\mathcal{P}\oplus\mathcal{P}^2\oplus\cdots\oplus\mathcal{P}^m$ are not 0, then this graph is connected and the element *(i,j)* gives all edge-chains from the vertex i to vertex j.

Corollary 3. Let $\mathcal{P}$ be the initial edge-chain matrix of a graph. There exists an edge-chain cycle with length k if and only if there are some diagonal elements of the matrix $\mathcal{P}^k$ are not 0.

From Corollary 3, if we calculate all edge-chain cycles with length k, just then calculate the diagonal of the matrix $\mathcal{P}^k$. And if need to calculate all edge-chain cycles with different lengths, then calculate the diagonal of matrix $\sum_{k=1}^{l}\mathcal{P}^k$.

Corollary 4. A graph has Eulerian cycles or paths if and only if the power length of $\mathcal{P}$ is m. It has Eulerian cycles if and only if all diagonal elements in matrix $\mathcal{P}^m$ are all with length m. It has an Eulerian paths if and only if some of the non-diagonal elements in matrix $\mathcal{P}^m$ are with length m. In other words, when the power length of $\mathcal{P}$ is m, if the diagonal elements in matrix $\mathcal{P}^m$ are not 0, then they are associated with all Eulerian cycles; and if some of the non-diagonal elements in matrix $\mathcal{P}^m$ are not 0, then they are associated with some Eulerian paths.

Definition 13: If the product of two edge-chains f_{ij} and g_{ji} $(i\neq j)$ is an Eulerian cycle, then f_{ij} and g_{ji} are called a pair of inverse elements.

Theorem 2: If the initial edge-chain matrix of a graph is $\mathcal{P}$, then the necessary and sufficient condition for the existence of the Eulerian cycle is that there exists at least a pair of inverse element in the matrix $\mathcal{P}^{m-1/2}$ and $\mathcal{P}^{m+1/2}$ (when n is an odd number) or $\mathcal{P}^{m/2}$ and itself (when n is an even number).

Proof: If there exists a Eulerian cycle, assume the edge-chain is $j_1 j_2 j_3\cdots j_{m-1}j_m$ ($j_1 j_2 j_3\cdots j_{m-1}j_m$ is a permutation of *1,2,3,…,m*).

If m is an odd, then $j_1 j_2 j_3 \cdots j_{m-1} j_m = j_1 j_2 j_3 \cdots j_{m-1/2} \otimes j_{m+1/2} \cdots j_{m-1} j_m$, where the length of the edge-chain $j_1 j_2 j_3 \cdots j_{m-1/2}$ is $m-1/2$ and it is in the matrix $\mathcal{P}^{m-1/2}$; the length of the edge-chain $j_{m+1/2} \cdots j_{m-1} j_m$ is $m+1/2$ and it is in the matrix $\mathcal{P}^{m+1/2}$. These two edge-chains are a pair of inverse elements.

If m is an even, then $j_1 j_2 j_3 \cdots j_{m-1} j_m = j_1 j_2 j_3 \cdots j_{m/2} \otimes j_{m+2/2} \cdots j_{m-1} j_m$, where the length of edge-chain $j_1 j_2 j_3 \cdots j_{m/2}$ is $m/2$ and it is in the matrix $\mathcal{P}^{m/2}$; and the length of the edge-chain $j_{m+2/2} \cdots j_{m-1} j_m$ is also $m/2$ and it is also in matrix $\mathcal{P}^{m/2}$. These two elements are a pair of inverse elements.

Conversely, if there exists at least a pair of inverse elements in those two matrixes, then the product of these two edge-chains represents an Eulerian cycle. By property (4), $\mathcal{P}^m = \mathcal{P}^{m-1/2} \otimes \mathcal{P}^{m+1/2}$ (when n is an odd number) or $\mathcal{P}^m = \mathcal{P}^{m/2} \otimes \mathcal{P}^{m/2}$ (when n is an odd number), thus there exists Eulerian cycle in this graph.□

If we note the first row of $\mathcal{P}$ as $\mathcal{P}_1^{(1)}$ and define $\mathcal{P}_1^{(k)} = \mathcal{P}_1^{(k-1)} \otimes \mathcal{P}$ $(k > 1)$, then $\mathcal{P}_1^{(k)}$ represents the first row of the matrix $\mathcal{P}^k$. And if $\mathcal{Q}_1^{(k)}$ represents the first column of the matrix $\mathcal{P}^k$, by property (4), we have $\mathcal{Q}_1^{(k)} = \mathcal{P} \otimes \mathcal{Q}_1^{(k-1)} = \mathcal{P}^{k-1} \otimes \mathcal{Q}_1^{(1)}$.

Corollary 5. If the initial edge-chain matrix of a graph is $\mathcal{P}$ and there exists an Eulerian cycle in this graph, then the product $\mathcal{P}_1^{(m-1/2)} \otimes \mathcal{Q}_1^{(m+1/2)}$ (or $\mathcal{P}_1^{(m+1/2)} \otimes \mathcal{Q}_1^{(m-1/2)}$ when n is an odd) or $\mathcal{P}_1^{(m/2)} \otimes \mathcal{Q}_1^{(m/2)}$ (when n is an even) represents all Eulerian cycles.

For undirected graphs, the element $(1, j)$ is the inverse order of the element $(j, 1)$ in any edge-chain matrix, so the calculation of the matrix $\mathcal{Q}_1^{(m+1/2)}$ and $\mathcal{Q}_1^{(m/2)}$ can be simplified by using $\mathcal{P}_1^{(m+1/2)}$ and $\mathcal{P}_1^{(m/2)}$ respectively.

5. Conclusion

We present two new concepts in this paper, the initial edge-chain matrix and the general edge-chain matrix, and then define some operations of these matrices to find all Eulerian cycles. This method can determine whether Eulerianian cycles exist or not and if they do can also find out all of them. Only through some power operations of the initial edge-chain matrix, can reveal all Eulerianian cycles which are showed in the final edge-chain matrix. This algorithm can be improved by computations of some row vectors and column vectors of some power of the initial edge-chain matrix. It is effective to directed or undirected finite graph.

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